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1991 J. Phys.: Condens. Matter 3 2441

(http://iopscience.iop.org/0953-8984/3/14/023)

J. Phys.: Condens. Matter 3 (1991) 2441–2449. Printed in the UK

COMMENT

The network model and its relation to classical percolation in quantum Hall systems

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Received 27 September 1990

Abstract. J T Chalker and P D Coddington proposed a model for studying the influence of quantum effects on the classical percolation transition in quantum Hall systems. A quantitative microscopic interpretation of the model is given, and the class of potentials which are described by the model is determined. An approximate calculation of the localization length proves that the classical localization–delocalization transition is not incorporated in the model.

1. Introduction

Percolation provides an intuitive picture for localization of electronic states in (integer) quantum Hall systems. In a smooth two-dimensional random potential and strong perpendicular magnetic field, an electron follows equipotential lines [1]. For a symmetric random potential, the typical diameter of an equipotential line diverges only at an energy equal to the average potential $\langle V(x, y)\rangle$ [2]. Therefore, extended electronic states exist only at the centre of each Landau band. The critical exponent of the diverging diameter of equipotential lines as function of energy has been calculated to $\nu = 4/3$ (see [3] and references therein).

The classical percolation model for localization in quantum Hall systems is strictly valid only for vanishing magnetic length due to the neglect of any interference between electronic states on different equipotential lines. The most important correction for finite magnetic length is tunnelling near saddle points between equipotential lines of the same energy. Although quantum tunnelling has no significant influence in the tails of a Landau band, it is expected to have consequences near the band centre due to the nearly-extended nature of electronic states there. Especially, it should alter characteristic features of the localization–delocalization transition, which is of principal interest for the understanding of the quantum Hall effect.

Despite the importance of quantum corrections for the classical percolation transition, there have been only a few attempts to determine corrections to the percolation model. The only proposition published until now which allows quantitative (numerical) calculations is a network model introduced by J T Chalker and P D Coddington [4]. Their model is built up of two elements: nodes and links, which connect the nodes in such a way that a square network is formed. Each link is characterized by a randomly chosen phase. Nodes are described by a two-dimensional transfer matrix

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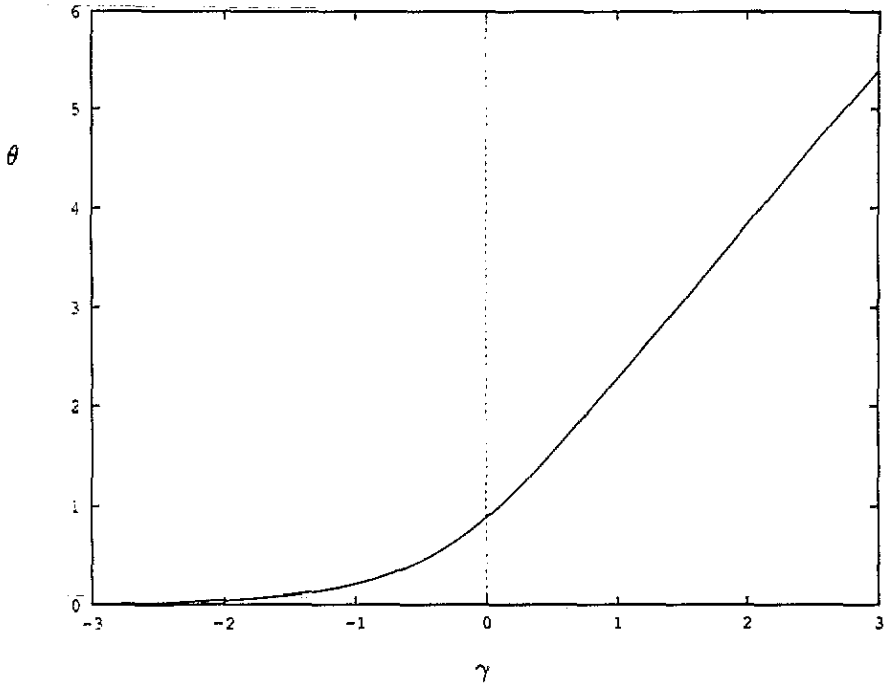

Figure 1. The θ parameter in the network model as a function of the reduced energy.

which is chosen from a one-parameter family; the corresponding parameter is called θ. Due to the different orientation of the current flow along the asymptotes, nodes in one out of two subslices have to be characterized in a similar way by a θ' parameter (see figure 3 in [4]). θ and θ' are connected by equation (6) in [4] for the symmetric version of the network model, which is invariant (on average) under rotation through 90°. To simplify the calculations, θ (or rather θ') has the same value at each node.

The θ parameter and the phases describing the links have been introduced into the network model solely by current conservation arguments. In the following section, a microscopic interpretation of the model parameters is given; thereby, the typical shape of a potential described by the model is deduced. The potential exhibits certain characteristic features of a periodic potential. As a consequence, the classical percolation transition is not incorporated in the network model. This statement is affirmed by approximately calculating the localization length in section 3. The calculation shows that the localization–delocalization transition in the network model is driven solely by backscattering effects. In the final section the comments on the model introduced by J T Chalker and P D Coddington are summarized briefly.

2. Microscopic interpretation of the network model

H A Fertig developed a WKB-type approximation for an electron moving in a smooth potential and high magnetic field [5]†. In the high-field limit

$$\|\nabla V\| \ll \hbar\omega_c/l_c \propto B^{3/2} \tag{1}$$

the cyclotron and centre coordinates describing the motion of an electron decouple; the

† A complete and rather straightforward proof of the approximation is given in [6].

centre coordinates follow equipotential lines except near saddle points where H A Fertig calculated the two-dimensional transfer matrix (see equation (2.11) in [5]; note that the definition of the transfer matrix slightly differs from [4]). By moving along an equipotential line the electron picks up an additional phase equal to the number of flux quanta enclosed between this equipotential line and the x-axis multiplied by 2π (see equation (2.6) in [5]). Each turning point—where the partial derivative of the potential with respect to y changes its sign and the partial derivative with respect to x is non-zero—contributes an additional term $\pi/2$ (see equation (2.7) in [5])†.

Using these approximate results, the network model is identified as a projection to one Landau level with the cyclotron and centre coordinates of the electronic motion being decoupled. The links correspond to isolated parts of equipotential lines, the nodes to saddle points.

The potential near a saddle point can be expanded—after an appropriate isometric transformation of the coordinate system—in the form:

$$V_{SP}(x, y) = -U_x x^2 + U_y y^2 + V_0. \tag{2}$$

Defining the reduced energy with respect to the saddle point by

$$\gamma := [E - \hbar\omega_c(n + \tfrac{1}{2}) - V_0]/\sqrt{U_x U_y}\, l_c^2 \tag{3}$$

where n stands for the Landau level index, one obtains for the energy-dependence of the θ and θ' parameter (figure 1):

$$\theta(E) = \operatorname{arcosh}(\sqrt{1 + e^{+\pi\gamma(E)}}) \tag{4a}$$

$$\theta'(E) = \operatorname{arcosh}(\sqrt{1 + e^{-\pi\gamma(E)}}). \tag{4b}$$

If one considers the symmetric version of the model, the reduced energy is the same in both formulae. Since θ is a strictly monotonic increasing function of energy, it can be approximated linearly near the band centre. This justifies the identification assumed in [4] of critical exponents on the θ scale with 'normal' critical exponents on the energy scale. In the limit of an infinitely high magnetic field, $\theta = \theta(E)$ assumes only two values—$\theta = 0$ and $\theta = \infty$—which corresponds to the absence of tunnelling in the classical limit.

Fixing the θ parameter (or rather θ') to the same value for all nodes means considering saddle points with identical values of V_0, which sets the effective zero of the energy scale. As the underlying potential should be symmetric to the Landau band centre, the common zero-point energy of the saddle points has to vanish. Consequently the network model incorporates quantum tunnelling only near saddle points with a zero-point energy at the classical percolation threshold in the band centre. Moreover, the product of the curvature parameters $U_x U_y$ is chosen to be identical for all saddle points.

The phase factors which characterize the links in the network model consist of two parts: the complex scattering matrices at the adjacent saddle points contribute a phase‡ and, secondly, an electron picks up a phase while travelling along an equipotential line. Isolated turning points do not need to be considered since they always occur in pairs of opposite curvature due to the topological set out of the network and therefore, contribute no additional phase.

† For closed equipotential lines the dependence of the connection formulae on the choice of the coordinate system drops out of the calculations.

‡ It is not possible to set this part of the phase factor to zero by a global gauge transformation, as was stated in [4].

Following the notation of equation (9) in [4] and denoting the phases characterizing the links in the following way

$$(A_j)_{nm} = \delta_{nm}\, e^{i\alpha_m(j)} \qquad (n, m \in \{1, \ldots, M\}) \tag{5a}$$

$$(C_j)_{nm} = \delta_{nm}\, e^{i\gamma_m(j)} \qquad (n, m \in \{1, \ldots, M\}) \tag{5b}$$

the calculation of the phase factors yields in cylindrical geometry:

$$\alpha_m(j) = -2\pi\mathcal{N}^{\Phi}(\gamma) + \pi/2 \qquad m \in \{1, 3, \ldots, M-1\} \tag{6a}$$

$$\alpha_m(j) = -2\pi\mathcal{N}^{\Phi}(\gamma) - 2\Phi(\gamma) - \pi \qquad m \in \{2, 4, \ldots, M\} \tag{6b}$$

$$\gamma_m(j) = -2\pi\mathcal{N}^{\Phi}(\gamma) + 2\Phi(\gamma) \qquad m \in \{1, 3, \ldots, M-1\} \tag{6c}$$

$$\gamma_m(j) = -2\pi\mathcal{N}^{\Phi}(\gamma) + \pi/2 \qquad m \in \{2, 4, \ldots, M\}. \tag{6d}$$

Here $\Phi(\gamma)$ is defined by

$$\Phi(\gamma) := \gamma/2 + \arg\Gamma(\tfrac{1}{2} + i\gamma/2) - (\gamma/2)\ln(|\gamma|/2) \tag{7}$$

and $\mathcal{N}^{\Phi}$ denotes the number of flux quanta enclosed by the equipotential line between the two adjacent saddle points—more precisely between the corresponding classical turning points at the considered energy—and the x-axis†. In strip geometry, there is one θ-type saddle point less per slice both at the lower and upper boundary of the network so that the phases at the boundaries are different:

$$\alpha_1(j) + \gamma_1(j-1) = -2\pi\mathcal{N}^{\Phi}(\gamma) + \Phi(\gamma) \tag{8a}$$

$$\alpha_M(j) + \gamma_M(j-1) = -2\pi\mathcal{N}^{\Phi}(\gamma) - \Phi(\gamma) \tag{8b}$$

where $\mathcal{N}^{\Phi}$ now denotes the flux quanta enclosed between two θ'-type saddle points in adjacent slices. As the phase shift due to scattering at saddle points is a given function of energy, the area of the 'unit cells' of the network has to vary randomly to allow for the random distribution of phase factors characterizing the links.

The microscopic interpretation of the model parameters shows that the network model corresponds to a potential which exhibits a square network of (nearly) identical saddle points with a certain freedom in choosing the curvature parameters and the relative orientation of the asymptotes of adjacent saddle points. The square network is distorted in such a way as to give rise to a random variation of the area of each unit cell. The average distance between two saddle points is retained as the length scale in the model, as the definition of the localization length shows (equation (14) in [4]).

In principle, it would be possible to include higher orders of the WKB-approximation in the given interpretation of the network model. This would give rise to correction terms in the expression for the reduced energy (3) and the phases (6)–(8); the determined energy dependence of the θ parameter (equation (4)) is exact if one inserts a generalized expression for the reduced energy [7]. The inclusion of higher-order corrections would not lead to any significant change in our understanding of the network model and, therefore, does not need to be considered.

The given microscopic interpretation of the model now enables us to answer the question of whether the underlying class of potentials is sufficiently general to allow one

† Integrated as usual from left to right, $\mathcal{N}^{\Phi}$ takes positive and negative values; the dependence on the considered link is not denoted explicitly.

to understand, by studying this model, the influence of quantum tunnelling on the localization–delocalization transition in a 'normal' smooth random potential.

An important simplifying feature of the network model is to incorporate quantum tunnelling only at saddle points with a zero-point energy at the band centre. This simplification, though, turns out to be not very restrictive. Since the tunnelling probability is exponentially small except for saddle points the energy of which coincides with that of the incoming wave (see equation (4)) and as there are only few saddle points† with a given zero-point energy, one expects quantum tunnelling to be important only near the classical percolation threshold. For sufficiently high magnetic fields, the leading correction to the classical percolation picture comes from saddle points at the band centre. The (small) fluctuation of zero-point energies of relevant saddle points should not lead to any significant correction.

A more restrictive feature of the model is that it contains a non-vanishing density of saddle points at the same zero-point energy. This is a characteristic property of periodic potentials and does not occur for a general random potential. As a consequence of this non-vanishing density, the diameter of a typical equipotential line is of the order of unity in the network model for all energies off the band centre; the percolating line at the band centre is not accompanied by a divergence of the average diameter of equipotential lines in its vicinity. Thus, the model does not contain the classical percolation transition. The network model considers finite equipotential lines, which couple via infinitely many saddle points, whereas it would be more appropriate for the localization–delocalization transition in a smooth random potential to take into account divergent equipotential lines, which couple via few saddle points.

As the density of saddle points at the band centre vanishes in the limit of infinite system size, it is in principle sufficient to consider only one saddle point. G V Mil'nikov and I M Sokolov [8] calculated for this the (quantum) contribution to the critical exponent of the localization length at the band centre to be $\Delta\nu = 1$. Adding to this the contribution due to the diverging average diameter of the equipotential lines, one gets for the critical exponent of the localization length $\nu = 7/3$. The agreement with the numerical data obtained in [4] and for a different model in [9] is remarkable and has been interpreted as evidence for the validity of the universality hypothesis.

The—certainly simplified—picture of divergent equipotential lines coupling via few saddle points assumes that quantum tunnelling can be treated as a correction to the classical percolation model and does not alter qualitatively the nature of the electronic states even in close vicinity to the band centre. Although this assumption needs further investigation, there is evidence from numerical simulations of the classical percolation model that saddle points play a less significant role than was expected [10]. For sufficiently large systems the percolating equipotential lines nearly always pass through one saddle point at most [11].

One may think of generalizing the network model by introducing a smeared-out distribution of zero-point energies at the nodes. According to the above considerations, its main effect would be to remove the non-vanishing density of saddle points at the same energy. Comparison of results obtained for different distributions and with the delta-peaked distribution of [4] may provide a reasonable test for universality in this context. If a calculation of the spatial distribution of eigenfunctions was possible for a distribution of zero-point energies which leads to a divergence of the equipotential lines at the band

† 'Few' means here that their density vanishes in the thermodynamic limit.

centre, this could be used to estimate the importance of quantum tunnelling for the percolation picture.

3. Delocalization due to backscattering

The microscopic interpretation of the network model has shown that it does not incorporate the classical percolation transition. This statement can be affirmed by an approximate calculation of the localization length which retains delocalization due to diverging equipotential lines, but neglects any backscattering and, thereby, quantum mechanical interference effects. The network model does not exhibit delocalization in this approximation.

It is advantageous to introduce the transmission coefficient of a quasi-one-dimensional network of width M normalized to the number of transmitting channels

$$T := (2/M)\,\mathrm{tr}\, tt^{\dagger} \tag{9}$$

where t is the transmission amplitude matrix. If $T_M(N)$ denotes the transmission coefficient of the network with width M and length N, one obtains for the localization length (following the convention of equation (14) in [4]):

$$\xi_M^{-1} = \lim_{N\to\infty} -(1/4N)\ln T_M(N) \tag{10}$$

using the fact that the transmission coefficient scales with the Lyapunov exponent of smallest absolute value. The localization length of the two-dimensional network model is then obtained by extrapolating the quasi-one-dimensional localization lengths for wide samples:

$$\xi = \lim_{M\to\infty} \xi_M . \tag{11}$$

The lowest order classical approximation for the scattering process is to replace the transmission coefficient of the network by the product of the transmission coefficients of the slices building up the network:

$$T_M(N) = \prod_{j=1}^{N} T_M^{(j)} . \tag{12}$$

This implies that any backscattering effects are neglected, i.e. paths are not taken into account along which the electron is reflected several times, but is finally transmitted through the system, and any interference effects are neglected. The replacement (12) is a good test for the occurrence of the percolation transition as the localization length calculated in this approximation diverges at energies where the average diameter of equipotential lines diverges itself. On the other hand, the approximate localization length remains finite if the percolating line passes through a non-vanishing number of saddle points per slice, which implies that the typical diameter of equipotential lines remains bounded in the vicinity of the corresponding energy.

Applying the approximation first to the one-dimensional network of width $M = 2$, one can calculate the transmission coefficient for a single slice by using the fact that the upper left quarter of the transfer matrix is the inverse of the transmission amplitude matrix if incoming and outgoing channels are numbered accordingly (for the transfer

matrix defining the network model see equations (9)–(12) in reference [4])†. In the approximation (12), one obtains for the localization length in cylindrical geometry

$$\xi_2 = 2/\ln(\cosh\theta' \cosh\theta) \tag{13}$$

since the sum of infinitely many randomly distributed phase-dependent terms can be replaced by a phase average. Equation (13) turns out to be an exact expression for the localization length of the one-dimensional network [12]. As the electronic states are extended over at most three 'unit cells' of the network, the cut-off of backscattering cannot influence the states significantly. It is even possible to get the exact result for the one-dimensional localization length without averaging, since (13) also follows if one extends the approximation (12) to the subslices of the network depicted in figure 3 of reference [4]. Nevertheless, for larger system widths it is necessary to consider at least one complete slice of the network as a unit to obtain an approximate localization length which does depend on the system width.

Using equations (9)–(12) of reference [4], one obtains for the transmission coefficient of a slice of the network of width M in cylindrical geometry:

$$T_M^{(j)} = \frac{1}{a^M + c^M + 2a^{M/2}c^{M/2}\cos\varphi(j)} \sum_{k=0}^{(M/2)-1} a^{2k}c^{2[(M/2)-1-k]} \tag{14}$$

where $\varphi(j)$ denotes an independent and uniformly distributed random phase variable for each slice j and

$$a := \cosh\theta' \cosh\theta \tag{15a}$$

$$c := \sinh\theta' \sinh\theta. \tag{15b}$$

After phase averaging one obtains in the approximation (12) for the localization length of a system of width M

$$\xi_M^{-1} = \tfrac{1}{4}\ln\{[a^m/(a^M - c^M)](a^2 - c^2)\}. \tag{16}$$

Therefore, the localization length of the two-dimensional network model without back scattering is in cylindrical geometry

$$\xi = 4/\ln[(\cosh\theta' \cosh\theta)^2 - (\sinh\theta' \sinh\theta)^2] \tag{17}$$

$$= 4/\ln(e^{\pi\gamma} + 1 + e^{-\pi\gamma}) \tag{18}$$

where the latter equality holds for the symmetric version of the model. A comparison of the two-dimensional localization length calculated without back scattering with the exact one-dimensional localization length (figure 2) shows the strong influence of back-scattering effects in the two-dimensional network model. These effects are solely responsible for the cross over from strongly localized states in the one-dimensional network to extended states in the two-dimensional network at the band centre.

According to (17), delocalized states in the direction of current flow considered here exist only if θ and θ' vanish simultaneously. This situation corresponds to classical percolating lines which force the electron, due to the strong asymmetry of the network, to propagate through the network from right to left—or left to right. The physically reasonable version of the network, which is symmetric under rotation through 90°, does

† Note that for $M > 2$ the incoming and outgoing channels have to be placed in an order which is different from that of reference [4] to obtain the convention for the transfer matrix used here.

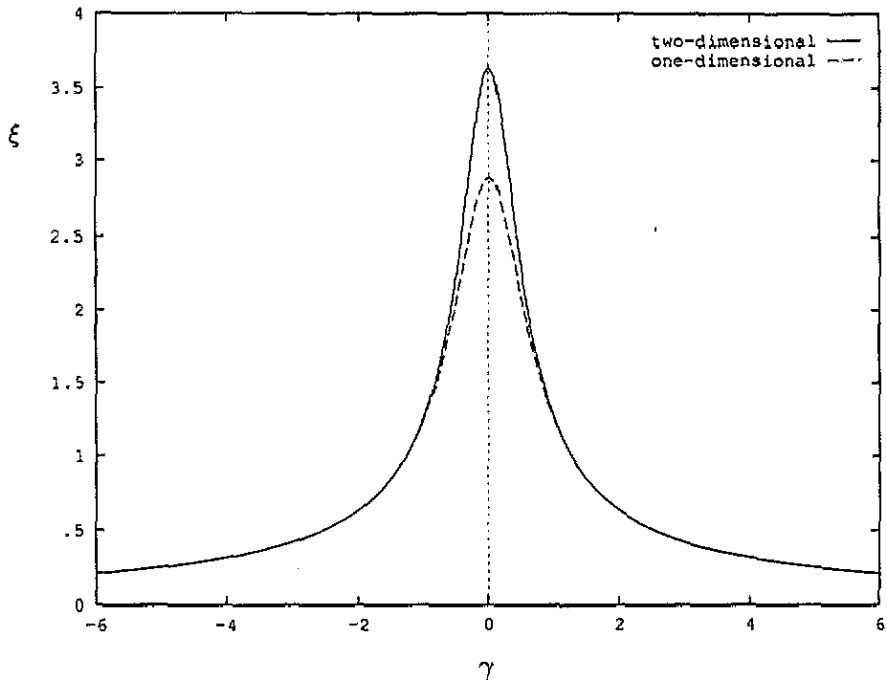

Figure 2. A comparison between the exact localization length in the one-dimensional network model and the approximate two-dimensional localization length calculated without backscattering.

not exhibit delocalized states in the approximation (12). Thus, the network model does not incorporate the classical percolation transition in a smooth random potential with high magnetic field.

4. Conclusions

A microscopic interpretation of the phenomenological parameters of the network model introduced by J T Chalker and P D Coddington has been given using a WKB-type approximation developed by H A Fertig for a smooth potential in high magnetic field. It has been possible to determine the characteristic features of potentials which are described by the model. Thereby it has been shown that the network model does not include the classical percolation transition; the localization–delocalization transition in this model is driven solely by backscattering effects. This statement has been affirmed by an approximate analytical calculation of the localization length where neglecting the backscattering leads to a strong localization of all electronic states.

Acknowledgments

I am greatly indebted to János Hajdu and Martin Janßen, who introduced me to this field, helped me through many long and stimulating discussions, and read the manuscript

of this paper critically. I would also like to thank John Chalker, who made some very valuable comments during the work.

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